Madras College Maths Department
Higher Maths
R&C 1.3 Differentiation

Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

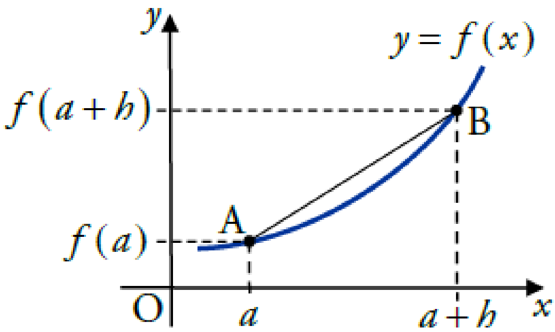
You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

Introduction to Differentiation RC

From our work on Straight Lines, we saw that the gradient (or "steepness") of a line is constant. However, the "steepness" of other curves may not be the same at all points.

In order to measure the "steepness" of other curves, we can use lines which give an increasingly good approximation to the curve at a particular point.

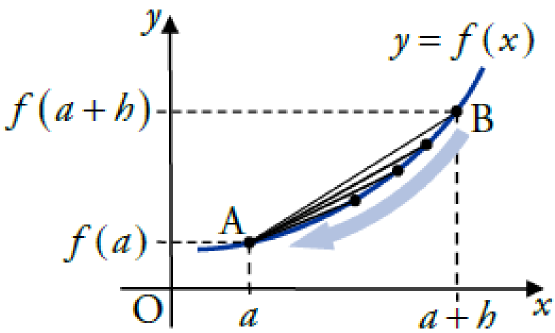
On the curve with equation $y = f(x)$, suppose point A has coordinates $(a, f(a))$.



At the point B where $x = a + h$, we have $y = f(a+h)$.

Thus the chord AB has gradient

$$m_{AB} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}.$$



If we let h get smaller and smaller, i.e. $h \to 0$, then B moves closer to A. This means that m_{AB} gives a better estimate of the "steepness" of the curve at the point A.

We use the notation $f'(a)$ for the "steepness" of the curve when $x = a$. So

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Given a curve with equation $y = f(x)$, an expression for $f'(x)$ is called the **derivative** and the process of finding this is called **differentiation.**

It is possible to use this definition directly to find derivates, but you will not be expected to do this. Instead, we will learn rules which allow us to quickly find derivatives for certain curves.

Finding the Derivative

The basic rule for differentiating $f(x) = x^n$, $n \in \mathbb{R}$, with respect to x is:

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Stated simply: the power (n) multiplies to the front of the x term, and the power lowers by one (giving $n-1$).

EXAMPLES

1. Given $f(x) = x^4$, find $f'(x)$.

2. Differentiate $f(x) = x^{-3}$, $x \neq 0$, with respect to x.

For an expression of the form $y = \ldots$, we denote the derivative with respect to x by $\frac{dy}{dx}$.

EXAMPLE

3. Differentiate $y = x^{-\frac{1}{3}}$, $x \neq 0$, with respect to x.

When finding the derivative of an expression with respect to x, we use the notation $\frac{d}{dx}$.

EXAMPLE

4. Find the derivative of $x^{\frac{3}{2}}$, $x \geq 0$, with respect to x.

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Terms with a coefficient

For any constant a,

$$\text{if } f(x) = a \times g(x) \text{ then } f'(x) = a \times g'(x).$$

Stated simply: constant coefficients are carried through when differentiating.

So if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$.

EXAMPLES

1. A function f is defined by $f(x) = 2x^3$. Find $f'(x)$.

2. Differentiate $y = 4x^{-2}$ with respect to x, where $x \neq 0$.

Differentiating more than one term

The following rule allows us to differentiate expressions with several terms.

If $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$.

Stated simply: differentiate each term separately.

EXAMPLES

1. A function f is defined for $x \in \mathbb{R}$ by $f(x) = 3x^3 - 2x^2 + 5x$.
   Find $f'(x)$.

2. Differentiate $y = 2x^4 - 4x^3 + 3x^2 + 6x + 2$ with respect to x.

Differentiating with Respect to Other Variables RC

So far we have differentiated functions and expressions with respect to *x*. However, the rules we have been using still apply if we differentiate with respect to any other variable. When modelling real-life problems we often use appropriate variable names, such as *t* for time and *V* for volume.

EXAMPLES

1. Differentiate $3t^2 - 2t$ with respect to *t*.

2. Given $A(r) = \pi r^2$, find $A'(r)$.

Remember
π is just a constant.

When differentiating with respect to a certain variable, all other variables are treated as constants.

EXAMPLE

3. Differentiate px^2 with respect to *p*.

Note
Since we are differentiating with respect to *p*, we treat x^2 as a constant.

Note

The derivative of an *x* term (e.g. $3x$, $\frac{1}{2}x$, $-\frac{3}{10}x$) is always a constant.

For example:

$$\frac{d}{dx}(6x) = 6, \qquad \frac{d}{dx}\left(-\frac{1}{2}x\right) = -\frac{1}{2}.$$

The derivative of a constant (e.g. 3, 20, π) is always zero.

For example:

$$\frac{d}{dx}(3) = 0, \qquad \frac{d}{dx}\left(-\frac{1}{3}\right) = 0.$$

Differentiable Form

Preparing to differentiate

It is important that before you differentiate, all brackets are multiplied out and there are no fractions with an x term in the denominator (bottom line).

For example:

$$\frac{1}{x^3}=x^{-3} \qquad \frac{3}{x^2}=3x^{-2} \qquad \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}} \qquad \frac{1}{4x^5}=\frac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2}=\frac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Differentiate $\sqrt{x}$ with respect to x, where $x>0$.

2. Given $y=\dfrac{1}{x^2}$, where $x\neq 0$, find $\dfrac{dy}{dx}$.

3. Differentiate $\dfrac{2}{x^3}$, $x\neq 0$, with respect to x.

Differentiating more complex expressions

We will now consider more complex examples where we will have to use several of the rules we have met.

EXAMPLES

1. Differentiate $y = \frac{1}{3x\sqrt{x}}$, $x > 0$, with respect to x.

Note
You need to be confident working with indices and fractions.

2. Find $\frac{dy}{dx}$ when $y = (x-3)(x+2)$.

Remember
Before differentiating, the brackets must be multiplied out.

3. A function f is defined for $x \neq 0$ by $f(x) = \frac{x}{5} + \frac{1}{x^2}$. Find $f'(x)$.

4. Differentiate $\frac{x^4 - 3x^2}{5x}$ with respect to x, where $x \neq 0$.

5. Differentiate $\frac{x^3+3x^2-6x}{\sqrt{x}}$, $x>0$, with respect to *x*.

Remember

$\frac{x^a}{x^b}=x^{a-b}$.

6. Find the derivative of $y=\sqrt{x}\left(x^2+\sqrt[3]{x}\right)$, $x>0$, with respect to *x*.

Remember

$x^a x^b=x^{a+b}$.

Rates of Change

R

The derivative of a function describes its "rate of change". This can be evaluated for specific values by substituting them into the derivative.

EXAMPLES

1. Given $f(x) = 2x^5$, find the rate of change of f when $x = 3$.

2. Given $y = \frac{1}{x^{\frac{2}{3}}}$ for $x \neq 0$, calculate the rate of change of y when $x = 8$.

Displacement, velocity and acceleration

The velocity v of an object is defined as the rate of change of displacement s with respect to time t. That is:

$$v = \frac{ds}{dt}.$$

Also, acceleration a is defined as the rate of change of velocity with respect to time:

$$a = \frac{dv}{dt}.$$

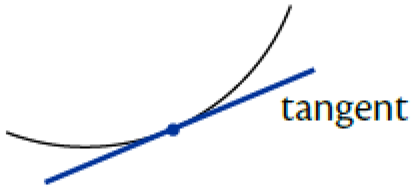
EXAMPLE

3. A ball is thrown so that its displacement s after t seconds is given by $s(t) = 12t - 5t^2$.

   Find its velocity after 2 seconds.

Equations of Tangents

As we already know, the gradient of a straight line is constant. We can determine the gradient of a curve, at a particular point, by considering a straight line which touches the curve at the point. This line is called a **tangent**.



The gradient of the tangent to a curve $y = f(x)$ at $x = a$ is given by $f'(a)$.

This is the same as finding the rate of change of f at a.

To work out the equation of a tangent we use $y - b = m(x - a)$. Therefore we need to know two things about the tangent:

- a point, of which at least one coordinate will be given;
- the gradient, which is calculated by differentiating and substituting in the value of x at the required point.

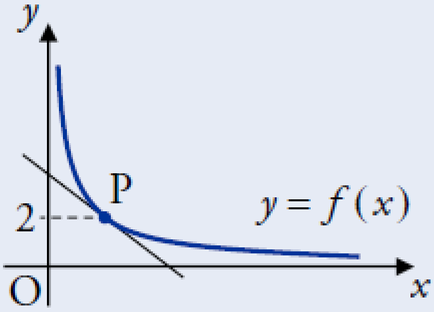
EXAMPLES

1. Find the equation of the tangent to the curve with equation $y = x^2 - 3$ at the point $(2, 1)$.

2. Find the equation of the tangent to the curve with equation $y = x^3 - 2x$ at the point where $x = -1$.

3. A function f is defined for $x > 0$ by $f(x) = \dfrac{1}{x}$.

Find the equation of the tangent to the curve $y = f(x)$ at P.



4. Find the equation of the tangent to the curve $y = \sqrt[3]{x}^2$ at the point where $x = -8$.

Differentiating sinx and cosx RC

In order to differentiate expressions involving trigonometric functions, we use the following rules:

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

These rules only work when x is an angle measured in radians. A form of these rules is given in the exam.

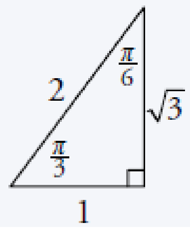
EXAMPLES

1. Differentiate $y = 3\sin x$ with respect to x.

2. A function f is defined by $f(x) = \sin x - 2\cos x$ for $x \in \mathbb{R}$.
   Find $f'\left(\frac{\pi}{3}\right)$.

Remember

The exact value triangle:



3. Find the equation of the tangent to the curve $y = \sin x$ when $x = \frac{\pi}{6}$.

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

The Chain Rule

We will now look at how to differentiate composite functions, such as $f(g(x))$. If the functions f and g are defined on suitable domains, then

$$\tfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \times g'(x).$$

Stated simply: differentiate the outer functions, the bracket stays the same, then multiply by the derivative of the bracket.

This is called the **chain rule**. You will need to remember it for the exam.

EXAMPLE

If $y = \cos\left(5x + \tfrac{\pi}{6}\right)$, find $\dfrac{dy}{dx}$.

Special Cases of the Chain Rule

We will now look at how the chain rule can be applied to particular types of expression.

Powers of a Function

For expressions of the form $\left[f(x)\right]^n$, where n is a constant, we can use a simpler version of the chain rule:

$$\tfrac{d}{dx}\left[\left(f(x)\right)^n\right] = n\left[f(x)\right]^{n-1} \times f'(x).$$

Stated simply: the power (n) multiplies to the front, the bracket stays the same, the power lowers by one (giving $n-1$) and everything is multiplied by the derivative of the bracket $\left(f'(x)\right)$.

EXAMPLES

1. A function f is defined on a suitable domain by $f(x)=\sqrt{2x^2+3x}$.
   Find $f'(x)$.

2. Differentiate $y=2\sin^4 x$ with respect to x.

EXAMPLES

3. Differentiate $y=(5x+2)^3$ with respect to x.

4. If $y = \frac{1}{(2x+6)^3}$, find $\frac{dy}{dx}$.

5. A function f is defined by $f(x) = \sqrt[3]{(3x-2)^4}$ for $x \in \mathbb{R}$. Find $f'(x)$.

Trigonometric Functions

The following rules can be used to differentiate trigonometric functions.

$$\frac{d}{dx}\left[\sin(ax+b)\right] = a\cos(ax+b), \qquad \frac{d}{dx}\left[\cos(ax+b)\right] = -a\sin(ax+b).$$

These are given in the exam.

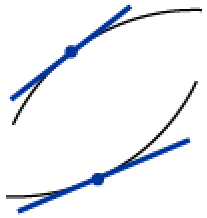
EXAMPLE

6. Differentiate $y = \sin(9x+\pi)$ with respect to x.

Increasing and Decreasing Curves

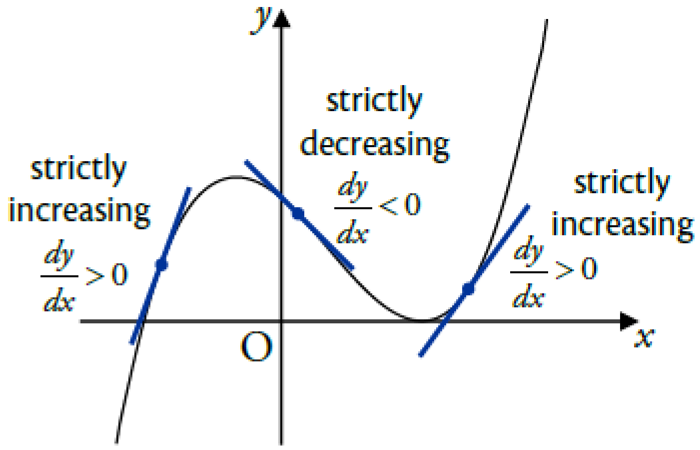
A curve is said to be **strictly increasing** when $\frac{dy}{dx} > 0$.

This is because when $\frac{dy}{dx} > 0$, tangents will slope upwards from left to right since their gradients are positive. This means the curve is also "moving upwards", i.e. strictly increasing.



Similarly:

A curve is said to be **strictly decreasing** when $\frac{dy}{dx} < 0$.



EXAMPLES

1. A curve has equation $y = 4x^2 + \frac{2}{\sqrt{x}}$.

   Determine whether the curve is increasing or decreasing at $x = 10$.

2. Show that the curve $y = \frac{1}{3}x^3 + x^2 + x - 4$ is never decreasing.

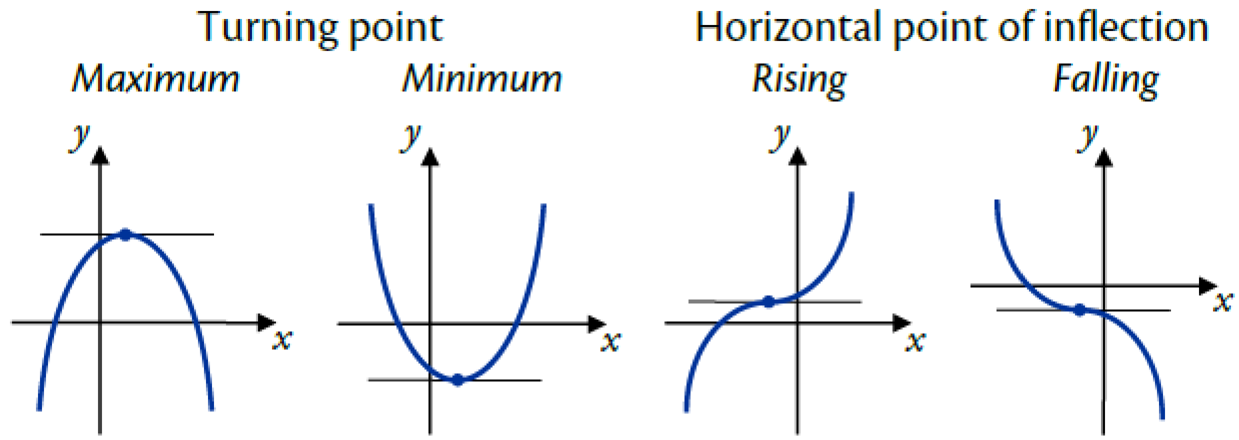
Remember
The result of squaring any number is always greater than, or equal to, zero.

Stationary Points RC

At some points, a curve may be neither increasing nor decreasing – we say that the curve is **stationary** at these points.

This means that the gradient of the tangent to the curve is zero at stationary points, so we can find them by solving $f'(x)=0$ or $\frac{dy}{dx}=0$.

The four possible stationary points are:



A stationary point's nature (type) is determined by the behaviour of the graph to its left and right. This is often done using a "nature table".

Determining the Nature of Stationary Points

To illustrate the method used to find stationary points and determine their nature, we will do this for the graph of $f(x) = 2x^3 - 9x^2 + 12x + 4$.

Step 1
Differentiate the function.

Step 2
Find the stationary values by solving $f'(x) = 0$.

Step 3
Find the y-coordinates of the stationary points.

Step 4
Write the stationary values in the top row of the nature table, with arrows leading in and out of them.

Step 5
Calculate $f'(x)$ for the values in the table, and record the results. This gives the gradient at these x values, so zeros confirm that stationary points exist here.

Step 6
Calculate $f'(x)$ for values slightly lower and higher than the stationary values and record the sign in the second row, e.g. $f'(0.8) > 0$ so enter $+$ in the first cell.

Step 7
We can now sketch the graph near the stationary points:
$+$ means the graph is increasing and
$-$ means the graph is decreasing.

Step 8
The nature of the stationary points can then be concluded from the sketch.

EXAMPLES

1. A curve has equation $y = x^3 - 6x^2 + 9x - 4$.

   Find the stationary points on the curve and determine their nature.

2. Find the stationary points of $y = 4x^3 - 2x^4$ and determine their nature.

3. A curve has equation $y = 2x + \frac{1}{x}$ for $x \neq 0$. Find the x-coordinates of the stationary points on the curve and determine their nature.

Curve Sketching

In order to sketch a curve, we first need to find the following:

- x-axis intercepts (roots) – solve $y = 0$;
- y-axis intercept – find y for $x = 0$;
- stationary points and their nature.

EXAMPLE

Sketch the curve with equation $y = 2x^3 - 3x^2$.

Example 2

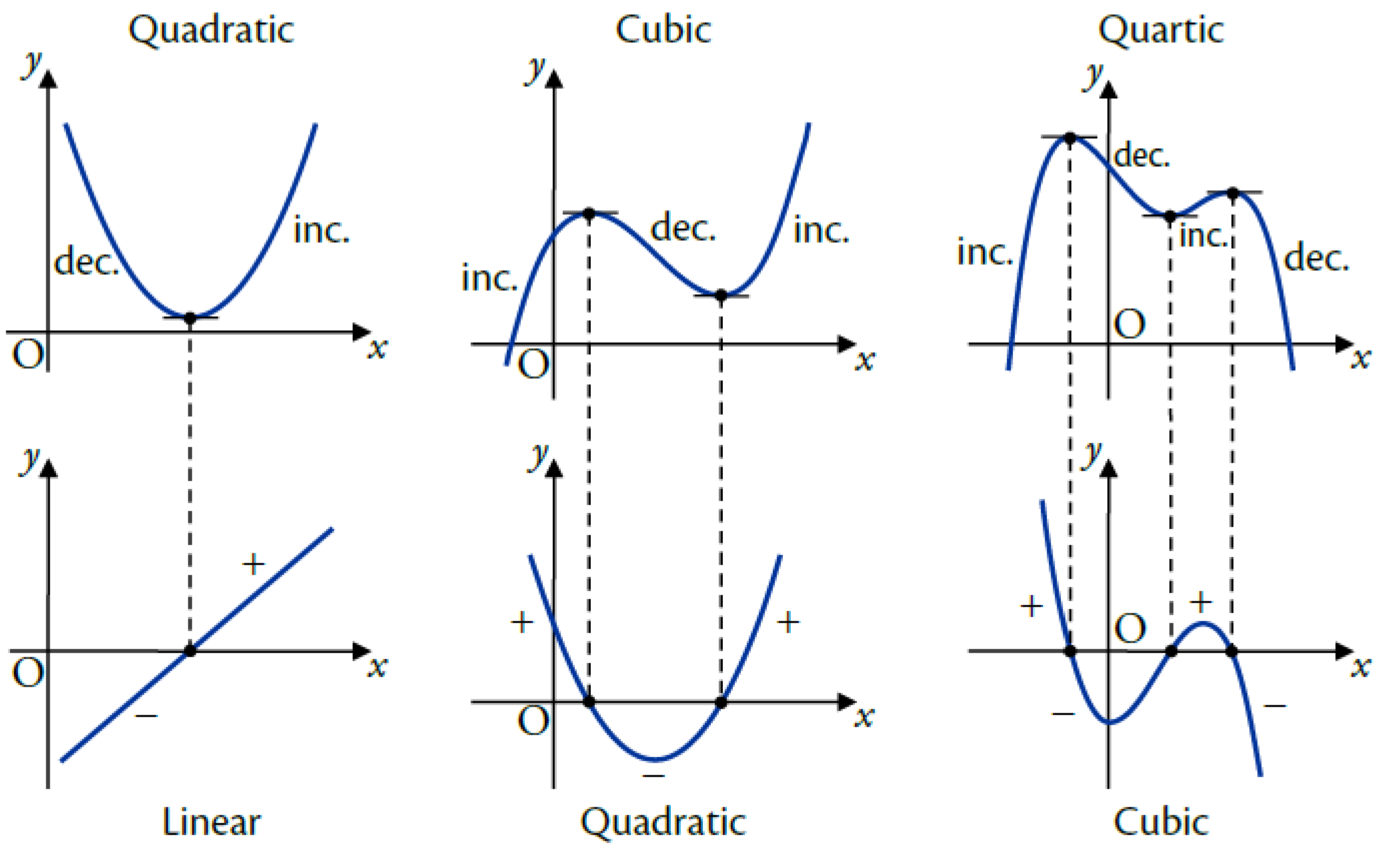
Sketch and annotate fully $y = x(x-3)^2$

Graphs of Derivatives EF

The derivative of an x^n term is an x^{n-1} term – the power lowers by one. For example, the derivative of a cubic (where x^3 is the highest power of x) is a quadratic (where x^2 is the highest power of x).

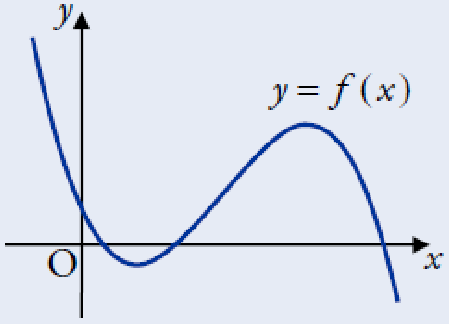
When drawing a derived graph:

- All stationary points of the original curve become roots (i.e. lie on the x-axis) on the graph of the derivative.
- Wherever the curve is strictly decreasing, the derivative is negative. So the graph of the derivative will lie below the x-axis – it will take negative values.
- Wherever the curve is strictly increasing, the derivative is positive. So the graph of the derivative will lie above the x-axis – it will take positive values.



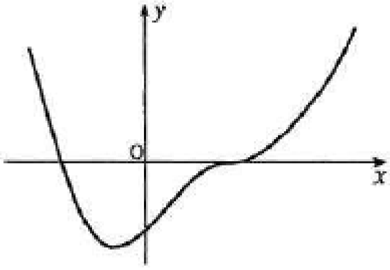
EXAMPLE

The curve $y = f(x)$ shown below is a cubic. It has stationary points where $x = 1$ and $x = 4$.



Sketch the graph of $y = f'(x)$.

Example 2 - The graph below has stationary points at (-1, -3) and (3, 0). Sketch the graph of $y = f'(x)$



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Practice Unit Assessments

Practice test 1

1 Find $f'(x)$, given that $f(x)=5\sqrt{x}-\frac{7}{x^3}$, $x>0$.

[3]

2 A bowler throws a cricket ball vertically upwards. The height (in metres) of the ball above the ground, t seconds after it is thrown, can be represented by the formula $h(t)=16t-4t^2$.

The velocity, v ms^{-1}, of the ball at time t is given by $v=\frac{dh}{dt}$.

Find the velocity of the cricket ball three seconds after it is thrown.
Explain what this means in the context of the question.

[#2.2 + 2]

3 Differentiate the function $f(x)=6\sin x$ with respect to x.

[1]

4 A curve has equation $y=7x^2+5x-3$.
Find the equation of the tangent to the curve at the point where $x=1$.

[4]

Practice test 2

1 Find $f'(x)$, given that $f(x)=x\sqrt[3]{x}+\frac{6}{\sqrt[4]{x^3}}$, $x>0$.

[3]

2 Differentiate $-3cosx$ With respect to x.

[1]

3 A particle moves in a horizontal line. The distance x (in metres) of the particle after t seconds can be represented by the formula $x=4t^2-24t$.

The velocity of the particle at time t is given by $v=\frac{dx}{dt}$.

(a) Find the velocity of the particle after three seconds.

[2]

(b) Explain your answer in terms of the particle's movement.

[#2.2]

4 A curve has equation $y=3x^2+2x-5$.
Find the equation of the tangent to the curve at the point where $x=-2$.

[4]

Homework

1

Given that $y = 12x^3 + 8\sqrt{x}$, where $x > 0$, find $\frac{dy}{dx}$. 3

SQA Higher Maths 2016 Non Calc Q2

2

(a) Find the x-coordinates of the stationary points on the graph with equation $y = f(x)$, where $f(x) = x^3 + 3x^2 - 24x$. **4**

(b) Hence determine the range of values of x for which the function f is strictly increasing. **2**

SQA Higher Maths 2016 Non Calc Q9

3

(a) Given that $y = \left(x^2 + 7\right)^{\frac{1}{2}}$, find $\frac{dy}{dx}$. **2**

(b) Hence find $\int \frac{4x}{\sqrt{x^2 + 7}}\,dx$. **1**

SQA Higher Maths 2016 Calc Q10

4

(a) Show that $\sin 2x \tan x = 1 - \cos 2x$, where $\frac{\pi}{2} < x < \frac{3\pi}{2}$. **4**

(b) Given that $f(x) = \sin 2x \tan x$, find $f'(x)$. **2**

SQA Higher Maths 2016 Calc Q11

5

Find the equation of the tangent to the curve $y = 2x^3 + 3$ at the point where $x = -2$. **4**

SQA Higher Maths 2015 Non Calc Q2

6

A function f is defined on a suitable domain by $f(x) = \sqrt{x}\left(3x - \frac{2}{x\sqrt{x}}\right)$.

Find $f'(4)$. **4**

SQA Higher
Maths 2015 Non -Calc Q7

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7

A curve has equation $y = 3x^2 - x^3$.

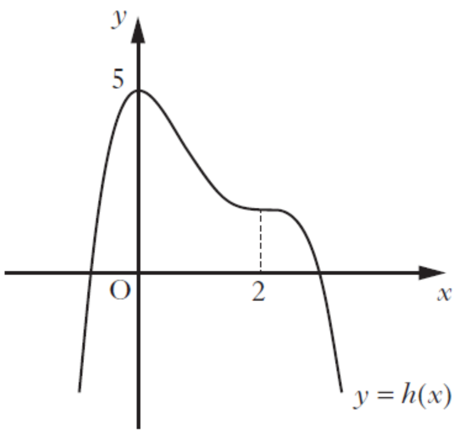
(*a*) Find the coordinates of the stationary points on this curve and determine their nature. **6**

(*b*) State the coordinates of the points where the curve meets the coordinate axes and sketch the curve. **2**

SQA Higher
Maths 2014 Non-Calc Q21

8

The diagram below shows the graph of a quartic $y = h(x)$, with stationary points at $x = 0$ and $x = 2$.



On separate diagrams sketch the graphs of:

(*a*) $y = h'(x)$; **3**

(*b*) $y = 2 - h'(x)$. **3**

SQA Higher
Maths 2012 Calc Q4

Practice Unit Assessment Solutions (1)

Differentiation – Practice Unit Assessment 1.

(1)

$f(x) = 5x^{\frac{1}{2}} - 7x^{-3}$ ✓

$f'(x) = \frac{1}{2} \times 5x^{-\frac{1}{2}} + 3 \times 7x^{-4}$ ✓

$= \frac{5x^{-\frac{1}{2}}}{2} + 21x^{-4}$ ✓

$= \frac{5}{2\sqrt{x}} + \frac{21}{x^4}$

(2) $v = h'(t) = 16 - 8t$ ✓

$v(3) = 16 - 8(3)$

$= -8\text{m/s}$ ✓

The ball is falling at 8m/s. ✓

(3) $f'(x) = 6\cos x$ ✓

(4) $y(1) = 7(1)^2 + 5(1) + 3$

$= 15$

$(1, 15)$

$\frac{dy}{dx} = 14x + 5$

$\frac{dy}{dx}(1) = 14(1) + 5$

$= 19$

$m = 19$

$y - b = m(x - a)$

$y - 15 = 19(x - 1)$

$y - 15 = 19x - 19$

$y = 19x - 4$

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Practice Unit Assessment Solutions (2)

① $f(x) = x\sqrt[3]{x} + \dfrac{6}{\sqrt[4]{x^3}}$ $\qquad \sqrt[3]{x} = x^{1/3}$, $\sqrt[4]{x^3} = x^{3/4}$

$f(x) = x \cdot x^{1/3} + \dfrac{6}{x^{3/4}}$

$f(x) = x^{4/3} + 6x^{-3/4}$

$f'(x) = \frac{4}{3}x^{+1/3} + -\frac{3}{4} \cdot 6x^{-7/4}$

$= \frac{4}{3}x^{1/3} - \frac{9}{2}x^{-7/4}$

$= \dfrac{4\sqrt[3]{x}}{3} - \dfrac{9}{2\sqrt[4]{x^7}}$

③ (a) $x = 4t^2 - 24t$

$v = \dfrac{dx}{dt} = 8t - 24$

$v(3) = 8(3) - 24$

$v(3) = 0$

(b) After 3 seconds the particle is instantaneously at rest

② $\dfrac{d}{dx}(-3\cos x)$

$= -3 \times -\sin x$

$= 3\sin x$

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4 $m = \frac{dy}{dx} = 6x + 2$

$m(-2) = 6(-2) + 2$

$m(-2) = -10$

$y(-2) = 3(-2)^2 + 2(-2) - 5$

$y(-2) = 3$

$m = -10$ at $(-2, 3)$

$y - b = m(x - a)$

$y - 3 = -10(x - (-2))$

$y - 3 = -10x - 20$

$\underline{y = -10x - 17}$